

# ESTIMATES FOR THE ERGODIC MEASURE AND POLYNOMIAL STABILITY OF PLANE STOCHASTIC CURVE SHORTENING FLOW

ABDELHADI ES-SARHIR, MAX-K. VON RENESSE, AND WILHELM STANNAT

**ABSTRACT.** We establish moment estimates for the invariant measure  $\mu$  of a stochastic partial differential equation describing motion by mean curvature flow in (1+1) dimension, leading to polynomial stability of the associated Markov semigroup. We also prove maximal dissipativity on  $L^1(\mu)$  for the related Kolmogorov operator.

## 1. INTRODUCTION AND PRELIMINARIES

We study the invariant measure  $\mu$  on  $L^2(0, 1)$  and the stability of the following SPDE for a function  $u(t) \in L^2(0, 1)$  introduced in [4], describing curve shortening flow in (1+1)D driven by additive noise

$$du(t) = (\arctan u_x(t))_x dt + \sigma dW_t, \quad t \geq 0. \quad (1.1)$$

Here  $W$  is cylindrical white noise on a separable Hilbert space  $U$  defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $\sigma$  is a Hilbert-Schmidt operator from  $U$  to the Sobolev space  $H_0^1(0, 1)$ . Existence of a unique generalized Markov solution of (1.1) and its ergodicity were shown in [4], working in the variational SPDE framework of Pardoux resp. Krylov-Rozovskiĭ. However, certain modifications of standard arguments apply since in contrast to previous works (like e.g. [2]) on variational SPDE the drift operator in (1.1) is neither coercive nor strongly dissipative. As a consequence exponential stability of the semigroup cannot be expected here, and it is our main goal to establish polynomial stability instead (see corollary 3.3 below). To this aim we derive moment estimates for the invariant measure of (1.1) which become crucial for the control of the contraction by the drift of (1.1) along the flow.

As a second application we establish the maximal dissipativity of the Kolmogorov operator  $J_0$  associated to (1.1), acting on smooth test functions  $\varphi : L^2(0, 1) \mapsto \mathbb{R}$  by

$$J_0 \varphi(u) = \frac{1}{2} \operatorname{Tr} Q D^2 \varphi(u) + \left\langle \frac{u_{xx}}{1 + u_x^2}, D\varphi(u) \right\rangle, \quad u \in D_0, \quad (1.2)$$

with the covariance operator  $Q = \sigma \sigma^*$  on  $L^2(0, 1)$  and

$$D_0 := \{u \in W_{loc}^{1,1}(0, 1) \mid (\arctan(u_x))_x \in L^2(0, 1)\}. \quad (1.3)$$

In contrast to the variational approach, here we shall work with a realization of the drift as a maximally monotone operator on  $L^2(0, 1)$  given by a subgradient  $V = \partial \Phi$  of a convex l.s.c. functional  $\Phi$  on  $L^2(0, 1)$ , using results of Andreu et al. [1] for variational PDE of linear growth functionals. Combining this with the moment estimates we prove that operator  $J_0$  defined on the domain  $D(J_0) = C_b^2(H) \subset L^1(H, \mu)$  with  $H = L^2(0, 1)$  is closable on  $L^1(H, \mu)$  and its closure generates a strongly continuous Markov semigroup on  $L^1(H, \mu)$  (cf. [7] for related results).

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2000 *Mathematics Subject Classification.* 47D07, 60H15, 35R60.

*Key words and phrases.* Degenerate stochastic equations, invariant measures, moment estimates.

The first two authors acknowledge support from the DFG Forschergruppe 718 "Analysis and Stochastics in Complex Physical Systems".

## 2. MOMENT ESTIMATES FOR THE INVARIANT MEASURE

In the sequel we denote by  $(e_k)_{k \geq 0}$  the system of eigenfunctions corresponding to the Laplace operator  $\Delta$  on  $(0, 1)$  with Dirichlet boundary condition. For  $n \geq 1$  we denote by  $H_n := \text{span}\{e_1, \dots, e_n\}$  and  $E := H_0^1(0, 1)$  and hence  $E^* = H^{-1}(0, 1)$ . Recall also that  $u \in L_{loc}^1(0, 1)$  belongs to the space  $BV$  of bounded variation functions if

$$[Du] := \sup \left\{ \int_{[0,1]} uv_x d\xi : v \in C_0^\infty(0, 1), \|v\|_\infty \leq 1 \right\} < +\infty.$$

The main result of this section reads as follows.

**Theorem 2.1.** *The measure  $\mu$  is concentrated on the subset  $D_0 \cap \{u \in L^2(0, 1) \mid u_x \in BV(0, 1)\}$  and*

$$\int [Du_x]^{\frac{1}{2}} \mu(du) + \int \|u\|_E^{\frac{1}{2}} \mu(du) + \int \|(\arctan u_x)_x\|_{L^2(0,1)}^2 \mu(du) < +\infty.$$

*Proof.* Introducing the operator  $A : E \rightarrow E^*$

$$\langle Au, v \rangle = - \int_0^1 \arctan u_x(z) \cdot v_x(z) dz, \quad u, v \in E.$$

we may write (1.1) as a variational SPDE in the Gelfand triple  $E \subset H \subset E^*$  as

$$du(t) = Au(t)dt + \sigma dW_t, \quad t \geq 0.$$

Below we write  $_{E^*}\langle \cdot, \cdot \rangle_E$  for the duality in  $E^* \times E$ , whereas  $\langle \cdot, \cdot \rangle_E$  denotes the inner product in  $E$ , i.e.  $_{E^*}\langle \xi, \zeta \rangle_E = \langle \xi, \zeta \rangle_{L^2(0,1)}$  and  $\langle \xi, \zeta \rangle_E = \langle \xi_x, \zeta_x \rangle_{L^2(0,1)}$  for  $\xi, \zeta \in C_c^\infty(0, 1)$ .

It is not difficult to see that the operator  $A$  satisfies the following properties.

(H1) For all  $u, v, x \in E$  the map

$$\mathbb{R} \ni \lambda \rightarrow_{E^*} \langle A(u + \lambda v), x \rangle_E$$

is continuous.

(H2) (Monotonicity) For all  $u, v \in E$

$$_{E^*}\langle Au - Av, u - v \rangle_E \leq 0.$$

(H3) For  $n \in \mathbb{N}$ , the operator  $A$  maps  $H^n := \text{span}\{e_1, \dots, e_n\} \subset E$  into  $E$  and there exists a constant  $c_1 \in \mathbb{R}$  such that

$$\langle Au, u \rangle_E + \|\sigma\|_{L^2(U, E)}^2 \leq c_1(1 + \|u\|_E^2) \quad \forall u \in H^n, n \in \mathbb{N}.$$

(H4) There exists a constant  $c_2 \in \mathbb{R}$  such that

$$\|A(u)\|_{E^*} \leq c_2(1 + \|u\|_E).$$

Define  $\mathcal{P}_n : E^* \rightarrow H_n$  by

$$\mathcal{P}_n y := \sum_{i=1}^n _{E^*}\langle y, e_i \rangle_E e_i, \quad y \in E^*.$$

Then  $\mathcal{P}_n|_H$  is just the orthogonal projection onto  $H_n$  in  $H$ . Define the family of  $n$ -dimensional Brownian motions in  $U$  by

$$W_t^n := \sum_{i=1}^n \langle W_t, f_i \rangle_U f_i = \sum_{i=1}^n B^i(t) f_i,$$

where  $(f_i)_{i \geq 1}$  is an orthonormal basis of the Hilbert space  $U$ . The  $n$ -dimensional SDE in  $H$

$$\begin{cases} du^n(t) = \mathcal{P}_n A u^n(t) dt + \mathcal{P}_n \sigma dW_t^n \\ u^n(0, x) = \mathcal{P}_n u_0(x) \end{cases} \quad (2.1)$$

may be identified with a corresponding SDE  $dx(t) = b^n(x(t))dt + \sigma^n(x(t))dB_t^n$  in  $\mathbb{R}^n$  via the isometric map  $\mathbb{R}^n \rightarrow H^n, x \rightarrow \sum_{i=1}^n x_i e_i$ . By [6, remark 4.1.2] conditions (H1) and (H2) imply the continuity of the fields  $x \rightarrow b^n(x) \in \mathbb{R}^n$ . Moreover, assumption (H2) implies

$$\langle b^n(x) - b^n(y), x - y \rangle_{\mathbb{R}^n} \leq c|x - y|^2, \quad \forall x, y \in \mathbb{R}^n$$

and, by the equivalence of norms on  $\mathbb{R}^n$ , (H3) gives the bound

$$\langle b^n(x), x \rangle + \|\sigma^n\|_{L_2(\mathbb{R}^n, \mathbb{R}^n)} \leq c(1 + |x|^2),$$

for some  $c > 0$ . Hence, equation (2.1) is a weakly monotone and coercive equation in  $\mathbb{R}^n$  which has a unique globally defined solution, cf. [6, chapter 3]. It is proved in [4] that for initial datum  $u_0 \in E$ , the process  $(u^n(t))_{t \geq 0}$  converges  $dt$ -a.e. in  $H$  to a process  $(u(t))_{t \geq 0}$ .

As in [4] we apply the Itô formula in finite dimensions to derive for  $t \rightarrow \|u^n(t)\|_E^2$

$$\begin{aligned} \|u^n(t)\|_E^2 &= \|u_0^n\|_E^2 + 2 \int_0^t \langle \mathcal{P}_n A(u^n(s)), u^n(s) \rangle_E ds + \int_0^t \|\mathcal{P}_n \sigma\|_{L_2(U_n, E)}^2 ds \\ &\quad + M^n(t), \quad t \in [0, T], \end{aligned}$$

where

$$M^n(t) := 2 \int_0^t \langle u^n(s), \mathcal{P}_n \sigma dW_s^n \rangle_E$$

and

$$\langle \mathcal{P}_n A(u^n(s)), u^n(s) \rangle_E = - \int_{(0,1)} \frac{(u_{xx}^n)^2}{1 + (u_x^n)^2} dx.$$

Taking expectation together with  $\|\mathcal{P}_n u_0(x)\|_E \leq \|u_0\|_E$  this entails

$$\frac{1}{t} \mathbb{E} \int_0^t \int_{(0,1)} \frac{(u_{xx}^n(s))^2}{1 + (u_x^n(s))^2} dx ds < C_1 \quad (2.2)$$

for some positive constant  $C_1$  independent of  $n$  and  $t$ . On the other hand, the Itô formula for  $\|u^n(t)\|_H^2$  reads

$$\begin{aligned} \|u^n(t)\|_H^2 &= \|u_0^n\|_H^2 + 2 \int_0^t \langle \mathcal{P}_n A(u^n(s)), u^n(s) \rangle_H ds + \int_0^t \|\mathcal{P}_n \sigma\|_{L_2(U_n, H)}^2 ds \\ &\quad + N^n(t), \quad t \in [0, T], \end{aligned} \quad (2.3)$$

with

$$N^n(t) := 2 \int_0^t \langle u^n(s), \mathcal{P}_n \sigma dW_s^n \rangle_H$$

and

$$\langle \mathcal{P}_n A(u^n(s)), u^n(s) \rangle_H = \int_{(0,1)} \frac{u_{xx}^n}{1 + (u_x^n)^2} u^n dx = - \int_{(0,1)} u_x^n \cdot \arctan(u_x^n) dx$$

Dividing by  $t$  and taking expectation in (2.3), using  $\arctan s \cdot s \geq |s| - K$  for some  $K > 0$  yield

$$\frac{1}{t} \mathbb{E} \int_0^t \int_{(0,1)} |u_x^n(s)| dx ds \leq C_2 \quad (2.4)$$

for some  $C_2 > 0$ . In particular, by the compactness of the embedding  $W_0^{1,1}(0,1) \subset L^2(0,1)$  for each  $n \in \mathbb{N}$  the family of measures  $\nu(n, t)(du) := \frac{1}{t} \int_0^t \mathbb{P}(u^n(s) \in du) ds$ ,  $t \geq 0$ , is tight

on  $L^2(0,1)$ . By analogous arguments as in [4] ergodicity of the Markov semigroup  $(P_t^n)_{t \geq 0}$  on  $L^2(0,1)$  associated to  $(u_t^n)_{t \geq 0}$  holds. Denoting by  $\nu_n$  the corresponding invariant distribution on  $L^2(0,1)$ , we may thus infer from (2.4), (2.2) and Birkhoff's ergodic theorem that for arbitrary  $L > 0$

$$\int \left( \int_{(0,1)} |u_x^n| dx \wedge L \right) \nu_n(du) + \int \left( \int_{(0,1)} \frac{u_{xx}^2}{1+u_x^2} dx \wedge L \right) \nu_n(du) < C$$

where  $C = C_1 + C_2$ . Letting tend  $L$  to infinity, by Fatou's lemma we obtain

$$\sup_{n \geq 1} \int \int_{(0,1)} |u_x| dx \nu_n(du) + \sup_{n \geq 1} \int \int_{(0,1)} \frac{u_{xx}^2}{1+u_x^2} dx \nu_n(du) < +\infty. \quad (2.5)$$

Since

$$\|(\arctan u_x)_x\|_{L^2(0,1)}^2 = \int_{(0,1)} \frac{u_{xx}^2}{(1+u_x^2)^2} dx \leq \int_{(0,1)} \frac{u_{xx}^2}{1+u_x^2} dx$$

this implies

$$\sup_{n \geq 1} \int_H \|(\arctan u_x)_x\|_{L^2(0,1)}^2 \nu_n(du) < +\infty. \quad (2.6)$$

Again, due to the compactness of  $W_0^{1,1}(0,1) \subset H$  the bound (2.5) implies that the sequence  $(\nu_n)_{n \geq 1}$  is tight w.r.t. the  $H$ -topology. This will now be amplified.

**Lemma 2.2.** *For  $u \in C_0^\infty(0,1)$*

$$\left( \int_{(0,1)} |u_{xx}(x)| dx \right)^{\frac{1}{2}} \leq \frac{1}{2} \int_{(0,1)} \frac{u_{xx}^2(x)}{1+u_x^2(x)} dx + \frac{3}{2} + \frac{1}{2} \|u_x\|_{L^1(0,1)}.$$

*Proof.* Starting from

$$\int_{(0,1)} |u_{xx}(x)| dx \leq \left( \int_{(0,1)} \frac{u_{xx}^2(x)}{1+u_x^2(x)} dx \right)^{\frac{1}{2}} \left( \int_{(0,1)} (1+u_x^2(x)) dx \right)^{\frac{1}{2}},$$

we get

$$\begin{aligned} \left( \int_{(0,1)} |u_{xx}(x)| dx \right)^{\frac{1}{2}} &\leq \left( \int_{(0,1)} \frac{u_{xx}^2(x)}{1+u_x^2(x)} dx \right)^{\frac{1}{4}} \left( \int_{(0,1)} (1+u_x^2(x)) dx \right)^{\frac{1}{4}} \\ &\leq \frac{1}{4} \int_{(0,1)} \frac{u_{xx}^2(x)}{1+u_x^2(x)} dx + \frac{3}{4} \left( \int_{(0,1)} (1+u_x^2(x)) dx \right)^{\frac{1}{3}}. \end{aligned}$$

Combining this with

$$\begin{aligned} \int_{(0,1)} (u_x(x))^2 dx &= - \int_{(0,1)} u_{xx}(x) u(x) dx \leq \int_{(0,1)} |u_{xx}(x)| dx \cdot \|u\|_\infty \\ &\leq \int_{(0,1)} |u_{xx}(x)| dx \cdot \|u_x\|_{L^1(0,1)} \end{aligned} \quad (2.7)$$

the claim is obtained using Youngs inequality

$$\begin{aligned} \left( \int_{(0,1)} |u_{xx}(x)| dx \right)^{\frac{1}{2}} &\leq \frac{1}{4} \int_{(0,1)} \frac{u_{xx}^2(x)}{1+u_x^2(x)} dx + \frac{3}{4} + \frac{3}{4} \left( \int_{(0,1)} |u_{xx}(x)| dx \right)^{\frac{1}{3}} \cdot \|u_x\|_{L^1(0,1)}^{\frac{1}{3}} \\ &\leq \frac{1}{4} \int_{(0,1)} \frac{u_{xx}^2(x)}{1+u_x^2(x)} dx + \frac{3}{4} + \frac{1}{2} \left( \int_H |u_{xx}(x)| dx \right)^{\frac{1}{2}} + \frac{1}{4} \|u_x\|_{L^1(0,1)}. \end{aligned}$$

□

Combining (2.5) with Lemma 2.2 we obtain a uniform bound

$$\sup_n \int_H \left( \int_{(0,1)} |u_{xx}(x)| dx \right)^{\frac{1}{2}} \nu_n(du) < \infty. \quad (2.8)$$

Due to the compactness of the embedding  $W^{2,1}(0,1) \hookrightarrow E$  this implies that the sequence of measures  $(\nu_n)_{n \geq 1}$  is tight w.r.t. the  $E$ -topology. Let  $\nu$  be the limit of a converging subsequence. From the weak convergence of  $\nu_n$  to  $\nu$  w.r.t. the  $E$ -topology and the fact that for  $\zeta \in L^2(0,1)$  the function  $u \rightarrow \langle \zeta, \arctan u_x \rangle_{L^2(0,1)}^2$  is bounded continuous on  $E$  we have

$$\int_H \langle e_k, \arctan u_x \rangle^2 \nu(du) = \lim_{n \rightarrow +\infty} \int_H \langle e_k, \arctan u_x \rangle^2 \nu_n(du).$$

Hence for  $m \geq 1$

$$\begin{aligned} \sum_{k=1}^m \int_H (\pi k)^2 \langle e_k, \arctan u_x \rangle^2 \nu(du) &= \lim_{n \rightarrow +\infty} \sum_{k=1}^m \int_H (\pi k)^2 \langle e_k, \arctan u_x \rangle^2 \nu_n(du) \\ &\leq \lim_{n \rightarrow +\infty} \int_H \sum_{k=1}^{+\infty} (\pi k)^2 \langle e_k, \arctan u_x \rangle^2 \nu_n(du) \\ &\leq \lim_{n \rightarrow +\infty} \int_H \|(\arctan u_x)_x\|_{L^2(0,1)}^2 \nu_n(du) < +\infty, \end{aligned}$$

using (2.6) in the last step. Sending  $m$  to infinity we arrive at

$$\int_H \|(\arctan u_x)_x\|_{L^2(0,1)}^2 \nu(du) = \sum_{k=1}^{+\infty} \int_H (\pi k)^2 \langle e_k, \arctan u_x \rangle^2 \nu(du) < +\infty.$$

Moreover, due to the lower semicontinuity of  $u \rightarrow [Du_x]$  w.r.t. to the  $E$ -topology (2.8) yields

$$\int [Du_x]^{\frac{1}{2}} \nu(du) < \infty.$$

From this and the boundedness of the embedding  $W_0^{2,1}(0,1)$  into  $W_0^{1,2}(0,1)$  we finally obtain

$$\int \|u\|_E^{\frac{1}{2}} \nu(du) < \infty.$$

It remains to show that the measures  $\nu$  and  $\mu$  coincide. Recall that for  $T > 0$  and regular initial condition  $u_0 \in E$  the sequence of Galerkin approximations  $u^n$  converges to  $u$  in the space  $L^2([0, T] \times \Omega, H)$ , c.f. [6, Chap. 4]. Hence, for all  $t > 0$ ,  $\rho > 0$ ,  $x \in E$  and bounded Lipschitz function  $\varphi : H \mapsto \mathbb{R}$

$$P_t^{n,\rho} \varphi(x) := \frac{1}{\rho} \int_t^{t+\rho} P_s^n \varphi(x) ds \longrightarrow P_t^\rho \varphi(x) := \frac{1}{\rho} \int_t^{t+\rho} P_s \varphi(x) ds.$$

A straightforward application of Itô's formula yields for all  $n \in \mathbb{N}$

$$|P_t^n \varphi(x) - P_t^n \varphi(y)| \leq \text{Lip}(\varphi) \|x - y\|_H \quad \forall x, y \in H.$$

Hence the family of functions  $(P_t^{n,\rho} \varphi)_{n \geq 0}$  is uniformly continuous on  $H$ , and for given compact subset  $K \subset H$  the Arzela-Ascoli theorem guarantees the existence of a subsequence of  $(P_t^{n,\rho} \varphi)_{n \geq 0}$  converging uniformly on  $K$  to  $P_t^\rho \varphi$ . Moreover, by (2.8) and Chebyshev's inequality for the collection of compact subsets  $K_R = \{u \in H \mid \|u\|_E \leq R\} \subset H$  we find

$$\lim_{R \rightarrow \infty} \sup_n \nu_n(H \setminus K_R) = 0.$$

These two facts allow to select a further subsequence, still denoted by  $n$ , such that

$$\lim_n \int_H P_t^{n,\rho} \varphi(x) \nu_n(dx) = \int_H P_t^\rho \varphi(x) \nu(dx).$$

Since  $\nu_n$  is  $P_t^n$ -invariant the l.h.s. above equals

$$\lim_n \int_H \varphi(x) \nu_n(dx) = \int_H \varphi(x) \nu(dx),$$

i.e.  $\nu$  is  $P_t^\rho$ -invariant, hence also  $P_t$ -invariant by letting  $\rho$  tend to zero. By the uniqueness of invariant measure for the ergodic semigroup  $(P_t)$  we conclude that  $\nu = \mu$ .  $\square$

### 3. POLYNOMIAL STABILITY

**Theorem 3.1.** *Let  $(u_t)_{t \geq 0}$ ,  $(v_t)_{t \geq 0}$  be two solutions of (1.1) with initial condition  $u_0, v_0 \in E$ . Then we have for  $\alpha \in (0, 1]$*

$$\|u_t - v_t\|_H^{2\alpha} \leq t^{-\alpha} \left( 3^\alpha \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right) \right) \|u_0 - v_0\|_H^{2\alpha}.$$

*Proof.* For the proof of the theorem we need the following elementary assertion.

**Lemma 3.2.** *For  $u, v \in E$  we have*

$$E^* \langle V(u) - V(v), u - v \rangle_E \leq - \frac{1}{\left( 1 + \|u\|_E^2 + \|v\|_E^2 \right)} \|u - v\|_H^2. \quad (3.1)$$

*Proof.* Let  $\gamma(t) := v + t(u - v)$ ,  $t \in [0, 1]$ . Then

$$\begin{aligned} E^* \langle V(u) - V(v), u - v \rangle_E &= - \int_{(0,1)} (\arctan u_x(r) - \arctan v_x(r)) (u_x(r) - v_x(r)) dr \\ &= - \int_{(0,1)} \int_{(0,1)} \frac{1}{1 + \gamma_x^2(t, r)} (u_x(r) - v_x(r))^2 dr dt, \end{aligned} \quad (3.2)$$

Note that for  $h := u - v$  we have  $h(0) = 0$  and hence for all  $t \in [0, 1]$

$$\begin{aligned} h^2(x) &= \left( \int_0^x h_x(r) dr \right)^2 \leq \int_0^x \frac{h_x^2(r)}{1 + \gamma_x^2(t, r)} dr \cdot \int_0^x (1 + \gamma_x^2(t, r)) dr \\ &\leq \int_0^x \frac{h_x^2(r)}{1 + \gamma_x^2(t, r)} dr \cdot \int_0^x (1 + u_x^2(r) + v_x^2(r)) dr \end{aligned}$$

which in view of (3.2) yields the claim after integration w.r.t.  $x$  and  $t$ .  $\square$

For  $u_0, v_0 \in E$  let  $(u_t)_{t \geq 0}$ ,  $(v_t)_{t \geq 0}$  be the strong solutions of (1.1) starting from  $u_0$  resp.  $v_0$ . Then

$$\frac{1}{2} \frac{d}{dt} \|u_t - v_t\|_H^2 = E^* \langle V(u_t) - V(v_t), u_t - v_t \rangle_E \leq - \frac{\|u_t - v_t\|_H^2}{1 + \|u_t\|_E^2 + \|v_t\|_E^2}.$$

In particular  $t \mapsto \|u_t - v_t\|_H^2$  is decreasing and thus

$$\begin{aligned} \|u_0 - v_0\|_H^2 &\geq \|u_t - v_t\|_H^2 + \int_0^t \frac{2\|u_s - v_s\|_H^2}{1 + \|u_s\|_E^2 + \|v_s\|_E^2} ds \\ &\geq \|u_t - v_t\|_H^2 \left( 1 + \int_0^t \frac{2}{1 + \|u_s\|_E^2 + \|v_s\|_E^2} ds \right). \end{aligned}$$

Since for any  $\alpha \in (0, 1]$  by Jensen's inequality

$$\left( 1 + t^{\alpha-1} \int_0^t \frac{2^\alpha}{\left( 1 + \|u_s\|_E^2 + \|v_s\|_E^2 \right)^\alpha} ds \right) \leq 2^{1-\alpha} \left( 1 + \int_0^t \frac{2}{1 + \|u_s\|_E^2 + \|v_s\|_E^2} ds \right)^\alpha$$

this implies

$$\|u_t - v_t\|_H^{2\alpha} \leq 2^{\alpha-1} \left( 1 + t^{\alpha-1} \int_0^t \frac{2^\alpha}{\left( 1 + \|u_s\|_E^2 + \|v_s\|_E^2 \right)^\alpha} ds \right)^{-1} \|u_0 - v_0\|_H^{2\alpha}. \quad (3.3)$$

Furthermore, using again Jensen for the convex function  $1/x$

$$\begin{aligned} \int_0^t \frac{1}{\left( 1 + \|u_s\|_E^2 + \|v_s\|_E^2 \right)^\alpha} ds &\geq \frac{t^2}{\int_0^t \left( 1 + \|u_s\|_E^2 + \|v_s\|_E^2 \right)^\alpha ds} \\ &\geq \frac{t^2}{3^{\alpha-1} \left( t + \int_0^t \|u_s\|_E^{2\alpha} ds + \int_0^t \|v_s\|_E^{2\alpha} ds \right)} \\ &= \frac{t}{3^{\alpha-1} \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right)}, \end{aligned}$$

which inserted into (3.3) gives

$$\begin{aligned} \|u_t - v_t\|_H^{2\alpha} &\leq 2^{\alpha-1} \left( 1 + \frac{2^\alpha t^\alpha}{3^{\alpha-1} \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right)} \right)^{-1} \|u_0 - v_0\|_H^{2\alpha} \\ &\leq 2^\alpha \left( 1 + \frac{2^\alpha t^\alpha}{3^\alpha \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right)} \right)^{-1} \|u_0 - v_0\|_H^{2\alpha} \\ &= 2^\alpha \frac{3^\alpha \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right)}{2^\alpha t^\alpha + 3^\alpha \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right)} \|u_0 - v_0\|_H^{2\alpha} \\ &\leq t^{-\alpha} \left( 3^\alpha \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{2\alpha} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{2\alpha} ds \right) \right) \|u_0 - v_0\|_H^{2\alpha}. \end{aligned}$$

□

Choosing  $\alpha = \frac{1}{4}$ , we obtain in particular

$$\|u_t - v_t\|_H^{\frac{1}{2}} \leq t^{-\frac{1}{4}} C \left( 1 + \frac{1}{t} \int_0^t \|u_s\|_E^{\frac{1}{2}} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{\frac{1}{2}} ds \right) \|u_0 - v_0\|_H^{\frac{1}{2}}, \quad (3.4)$$

for some positive constant  $C$ . As a consequence we arrive at the following statement.

**Corollary 3.3.** *Let  $\varphi : L^2(0, 1) \mapsto \mathbb{R}$  bounded and  $\frac{1}{2}$ -Hölder-continuous, i.e.*

$$\sup_{x \neq y \in L^2(0,1)} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|_{L^2(0,1)}^{1/2}} =: |\varphi|_{1/2} < \infty,$$

then for  $u, v \in E$

$$\limsup_{t \rightarrow \infty} \left[ t^{1/4} \frac{|P_t \varphi(u) - P_t \varphi(v)|}{\|u - v\|_{L^2(0,1)}^{1/2}} \right] \leq \frac{1}{\sqrt{2}} |\varphi|_{1/2} C \left( 1 + 2 \int \|u\|_E^{\frac{1}{2}} \mu(du) \right).$$

*Proof.* Using (3.4)

$$\begin{aligned} |P_t \varphi(u) - P_t \varphi(v)| &= |\mathbb{E}(\varphi(u_t) - \varphi(v_t))| \leq |\varphi|_{1/2} \mathbb{E}(\|u_t - v_t\|_H^{\frac{1}{2}}) \\ &\leq t^{-\frac{1}{4}} \|u - v\|_H^{\frac{1}{2}} \cdot C \left(1 + \mathbb{E}\left(\frac{1}{t} \int_0^t \|u_s\|_E^{\frac{1}{2}} ds + \frac{1}{t} \int_0^t \|v_s\|_E^{\frac{1}{2}} ds\right)\right), \end{aligned}$$

where  $\frac{1}{t} \mathbb{E} \int_0^t \|u_s\|_E^{\frac{1}{2}} ds$  converges to  $\int_H \|u\|_E^{\frac{1}{2}} \mu(du)$  as  $t \rightarrow +\infty$ , due to the ergodicity of  $(P_t)$ .  $\square$

#### 4. MAXIMAL DISSIPATIVITY OF THE OPERATOR $J$

In this final section we prove the maximal dissipativity of the operator  $(J_0, D(J_0))$  on the space  $L^1(H, \mu)$ , where  $J_0$  is defined in (1.3) and  $D(J_0) := C_b^2(H)$ . As a standard consequence the transition semigroup  $(P_t)$  corresponding to the generalized solution of (1.1) admits a unique extension to a strongly continuous semigroup  $(P_t^0)_{t \geq 0}$  on  $L^1(H, \mu)$ .

For the proof we exploit that the drift in (1.1) can be associated to a subdifferential of a convex l.s.c. functional on  $L^2(0, 1)$ , using the general set-up introduced in [1] for  $L^2$ -gradient flows of linear growth functionals. Let  $G$  denote the primitive of the function  $s \mapsto \arctan s$ , then  $G$  is a convex function with linear growth at infinity. For a measure  $\nu$  on  $[0, 1]$  with Lebesgue decomposition

$$\nu := hdx + \nu^s$$

with  $\nu = h|\nu|$  and  $\nu^s$  is singular part of  $\nu$ , we define a new measure  $G(\nu)$  on the Borel sets  $B \subset [0, 1]$  by

$$\int_B G(\nu) := \int_B G(h(x)) dx + \int_B G_\infty \left( \frac{d\nu}{d|\nu|} \right) d|\nu|^s.$$

where

$$G_\infty(x) := \lim_{t \rightarrow +\infty} \frac{G(tx)}{t} = \frac{\pi}{2} x.$$

We introduce the functional  $\Phi$  on  $L^2(0, 1)$

$$\Phi(x) = \begin{cases} \int_{[0,1]} G(Du), & u \in BV(0, 1) \\ +\infty, & u \in L^2(0, 1) \setminus BV(0, 1). \end{cases}$$

By the results in [1] the functional  $\Phi$  is convex on  $BV(0, 1)$  and lower semicontinuous on every  $L^p(0, 1)$ . Hence the subdifferential  $\partial\Phi$  of  $\Phi$ , which is the multi-valued operator in  $L^2(0, 1)$  defined by

$$v \in \partial\Phi \iff \Phi(\zeta) - \Phi(u) \geq \int_{(0,1)} v(\zeta - u) dx, \quad \forall \zeta \in L^2(0, 1)$$

is a maximal monotone operator in  $L^2(0, 1)$ . Clearly  $u \in BV_0^1(0, 1)$  if  $u \in W_0^{1,1}(0, 1)$  with  $\|Du\| = \|u_x\|_{L^1(0,1)}$  and so if

$$v = -(\arctan u_x)_x \in L^2(0, 1),$$

then  $v \in \partial\Phi(u)$ . Moreover, since

$$\text{For } \zeta \in \mathbb{R}, \quad |\zeta| - C_1 \leq \arctan \zeta \cdot \zeta \quad \text{for some } C_1 > 0$$

$u \in D_0$  implies  $u \in W^{1,1}(0, 1)$ , i.e. with  $V(u) := -\partial\Phi(u)$  for  $u \in D_0$  we find that  $V(u) = u_{xx}/(1 + u_x^2)$ . Thus, considering the Kolmogorov operator  $J_0$  as unbounded operator in  $L^1(H, \mu)$  with domain  $D(J_0) := C_b^2(H)$  we can write for  $\varphi \in C_b^2(H)$

$$J_0 \varphi(u) = \frac{1}{2} \text{Tr } QD^2 \varphi(u) + \langle V(u), D\varphi(u) \rangle, \quad \varphi \in C_b^2(H).$$



Note that this definition of  $J_0$  makes sense in  $L^1(H, \mu)$ , because by Theorem 2.1

$$\int_H V(u)^2 \mu(du) < +\infty.$$

Secondly, it follows from Itô's-formula for  $\|u(t)\|_H^2$  for solutions with regular initial condition that the measure  $\mu$  is infinitesimally invariant for the operator  $J_0$ , i.e.

$$\int J_0 \varphi(u) \mu(du) = 0 \quad \text{for all } \varphi \in D(J_0),$$

and moreover, since

$$J_0 \varphi^2 = 2\varphi J_0 \varphi + \frac{1}{2} \langle Q D\varphi, D\varphi \rangle, \quad \varphi \in D(J_0)$$

also

$$\int_H J\varphi(u) \varphi(u) \mu(du) = -\frac{1}{2} \int_H \|\sigma^* D\varphi(u)\|^2 \mu(du).$$

which entails that  $J_0$  is dissipative in the Hilbert space  $L^2(H, \mu)$ . By similar argument as in [5] one proves that  $J_0$  is also dissipative in  $L^1(H, \mu)$ . Therefore it is closable and its closure  $J := \bar{J}_0$  with domain  $D(J)$  is dissipative. Now the main assertion of this section reads as follows.

**Theorem 4.1.** *The operator  $(J, D(J))$  generates a  $C_0$ -semigroup of contractions on  $L^1(H, \mu)$ .*

*Proof.* We shall prove that  $\text{rg}(\lambda - J)$  is dense in  $L^1(H, \mu)$ . To this aim for  $\alpha > 0$  consider the Yosida approximation of  $V$  defined by

$$V_\alpha(x) = V(J_\alpha(x)), \quad \text{where } J_\alpha(x) = (\text{Id} - \alpha V)^{-1}(x), \quad x \in D(V).$$

For the sequence  $V_\alpha$  we have the following:

- (i) For any  $\alpha > 0$ ,  $V_\alpha$  is dissipative and Lipschitz continuous.
- (ii)  $\|V_\alpha(x)\| \leq \|V(x)\|$  for any  $x \in D(V)$ .

Note that the function  $V_\alpha$  is not differentiable in general. Therefore we shall consider a  $C^1$ -approximation as in [3]. For  $\alpha, \beta > 0$  we set

$$V_{\alpha,\beta}(x) := \int_H e^{\beta \Delta} V_\alpha(e^{\beta \Delta} x + y) \mathcal{N}_{0,\sigma_\beta}(dy)$$

where  $\mathcal{N}_{0,\sigma_\beta}$  is the Gaussian measure on  $H$  with mean 0 and covariance operator defined by  $\sigma_\beta := \int_0^\beta e^{2s\Delta} ds$ . Then,  $V_{\alpha,\beta}$  is dissipative and by the Cameron-Martin formula it is  $C^\infty$  differentiable. Moreover, as  $\alpha, \beta \rightarrow 0$ ,  $V_{\alpha,\beta} \rightarrow V$  pointwise. Let us now introduce the following approximating equation

$$\begin{cases} du_{\alpha,\beta}(t) = V_{\alpha,\beta}(u_{\alpha,\beta}(t))dt + \sigma dW_t, & t \geq 0 \\ u_{\alpha,\beta}(0) = x. \end{cases} \quad (4.1)$$

Since  $V_{\alpha,\beta}$  is globally Lipschitz, equation (4.1) has a unique strong solution  $(u_{\alpha,\beta}(t))_{t \geq 0}$ . Moreover by the regularity of  $V_{\alpha,\beta}$  the process  $(u_{\alpha,\beta}(t))_{t \geq 0}$  is differentiable on  $H$ . For any  $h \in H$  we set  $\eta_h(t, x) := Du_{\alpha,\beta}(t, x) \cdot h$  it holds

$$\begin{cases} \frac{d}{dt} \eta_h(t, x) = DV_{\alpha,\beta}(u_{\alpha,\beta}(t, x)) \cdot \eta_h(t, x), \\ \eta^h(0, x) = h \in H. \end{cases} \quad (4.2)$$

From the dissipativity of  $V_{\alpha,\beta}$  we have that

$$\langle DV_{\alpha,\beta}(z)h, h \rangle \leq 0, \quad h \in H, z \in D(V).$$

Hence by multiplying both sides of (4.2) by  $\eta_h(t, x)$ , integrating with respect to  $t$ , we have

$$\|\eta_h(t, x)\|^2 \leq \|h\|^2. \quad (4.3)$$

Now for  $\lambda > 0$  and  $f \in C_b^2(H)$ , consider the following elliptic equation

$$(\lambda - J_{V_{\alpha,\beta}})\varphi_{\alpha,\beta} = f, \quad \lambda > 0. \quad (4.4)$$

where  $J_{V_{\alpha,\beta}}$  is the Kolmogorov operator corresponding to the SDE (4.1). It is well-known that this equation has a solution  $\varphi_{\alpha,\beta} \in C_b^2(H)$  and can be written in the form  $\varphi_{\alpha,\beta} = R(\lambda, J_{V_{\alpha,\beta}})f$ , where

$$(R(\lambda, J_{V_{\alpha,\beta}})f)(x) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}(f(u_{\alpha,\beta}(t, x))) dt$$

is the pseudo resolvent associated with  $J_{V_{\alpha,\beta}}$ . Thus we have

$$\|\lambda \varphi_{\alpha,\beta}\|_\infty \leq \|f\|_\infty. \quad (4.5)$$

We have, moreover, for all  $h \in H$ ,

$$D\varphi_{\alpha,\beta}(x)h = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}\left(Df(u_{\alpha,\beta}(t, x))(Du_{\alpha,\beta}(t, x)h)\right) dt.$$

consequently by using (4.3) it follows that

$$\sup_{\alpha, \beta > 0} \|D\varphi_{\alpha,\beta}(x)\| \leq \frac{1}{\lambda} \|Df\|_\infty.$$

From (4.4) we have

$$\begin{aligned} \lambda \varphi_{\alpha,\beta}(x) - \frac{1}{2} \text{Tr } Q D^2 \varphi(x) + \langle V(x), D\varphi_{\alpha,\beta}(x) \rangle \\ = f(x) + \langle V(x) - V_{\alpha,\beta}(x), D\varphi_{\alpha,\beta}(x) \rangle, \quad \lambda > 0, \quad x \in D(V). \end{aligned}$$

Using gradient bound (4.5) we deduce that

$$\int_H |\langle V_{\alpha,\beta}(x) - V(x), D\varphi_{\alpha,\beta}(x) \rangle| \mu(dx) \leq \frac{1}{\lambda^2} \|Df\|_\infty^2 \|V_{\alpha,\beta} - V\|_{L^2(H, \mu)}.$$

By Lebesgue's theorem  $\|V_{\alpha,\beta} - V\|_{L^2(H, \mu)}$  converges to 0 as  $\alpha, \beta \rightarrow 0$ . Therefore we deduce that for  $\alpha, \beta \rightarrow 0$ ,

$$\lambda \varphi_{\alpha,\beta}(x) - \frac{1}{2} \text{Tr } Q D^2 \varphi_{\alpha,\beta}(x) + \langle V(x), D\varphi_{\alpha,\beta}(x) \rangle \rightarrow f$$

strongly in  $L^1(H, \mu)$ . This implies that

$$C_b^2(H) \subset \overline{(\lambda - J_0)(D(J_0))}.$$

Since  $C_b^2(H)$  is dense in  $L^1(H, \mu)$ , the proof is complete.  $\square$

## REFERENCES

- [1] F. Andreu, V. Caselles and J. M. Mazón, *A strongly degenerate quasilinear equation: the parabolic case*, Arch. Ration. Mech. Anal. **176** (2005), no. 3, 415–453.
- [2] V. Barbu and G. Da Prato, *Ergodicity for nonlinear stochastic equations in variational formulation*, Appl. Math. Optim. **53** (2006), no. 2, 121–139.
- [3] G. Da Prato, J. Zabczyk, *Second Order Partial Differential Equations in Hilbert Spaces*, London Mathematical Society Lecture Notes, vol. 283, Cambridge University Press, 2002.
- [4] A. Es-Sarhir and Max-K. von Renesse, *Ergodicity of stochastic curve shortening flow in the plane*, submitted (2010).
- [5] A. Es-Sarhir and W. Stannat, *Invariant measures for semilinear SPDE's with local Lipschitz drift coefficients and applications*, J. Evol. Equ. **8** (2008), 129–154.

- [6] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
- [7] W. Stannat, *(Nonsymmetric) Dirichlet Operators on  $L^1$  - Existence, Uniqueness and Associated Markov Processes*, Ann. Scuola Norm. Sup. Pisa, Cl.Sci (4) **28** (1999), 99–140.

TECHNISCHE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK  
STRASSE DES 17. JUNI 136, D-10623 BERLIN, GERMANY

TECHNISCHE UNIVERSITÄT DARMSTADT, FACHBEREICH MATHEMATIK,  
SCHLOSSGARTENSTRASSE 7, D-64289 DARMSTADT, GERMANY  
*E-mail address:* `stannat@mathematik.tu-darmstadt.de`  
*E-mail address:* `[essarhir,mrenesse]@math.tu-berlin.de`